Math 245B Lecture 20 Notes

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1 Introduction to L^p Spaces

1.1 L^p spaces and norms

Fix a measure space $(X\mathcal{M}, \mu)$. We will deal with complex functions, but the real case is the same.

Definition 1.1. Let $0 < p < \infty$, and let $f : X \to \mathbb{C}$ be measurable. The L^p **norm**[1] is

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

If f doesn't have a lot of spiky parts in its graph, then the L^p norm of f is about the value of f. When the graph has huge peaks, as p gets bigger, the spikes are amplified. Likewise, as p gets bigger, tails of functions are suppressed.

Definition 1.2. The L^p **space** $L^p(X, \mathcal{M}, \mu) = L^p(\mu) = L^p$ is the space of measurable functions $f : X \to \mathbb{C}$ such that $\|f\|_p < \infty$.

Example 1.1. Let X be a countable set with the measure μ, counting measure on $(X, \mathscr{P}(X))$. Then $\ell^p(X) := L^p(\mu)$. As an example,

$$\ell^p(\mathbb{N}) = \ell^p = \left\{ (x_n)_n \in \mathbb{C}^{\mathbb{N}} : \sum_n |x_n|^p < \infty \right\}.$$

Lemma 1.1. *For all $p \in (0, \infty)$, $L^p(\mu)$ is a vector space over $\mathbb{C}$.*

Proof. If $\|f\|_p, \|g\|_p < \infty$,

$$|f + g|^p \le (2 \max(|f|, |g|))^p = 2^p \max(|f|^p, |g|^p) \le 2^p(|f|^p + |g|^p).$$

So

$$\int |f + g|^p \, d\mu \le 2^p \int |f|^p + 2^p \int |g|^p < \infty.$$

□

[1] This is only really a norm when $p \ge 1$.

1.2 L^p norm inequalities

Now assume $p \geq 1$. We want to show that L^p is a normed space. These inequalities will help us, but they are very important to know on their own.

Lemma 1.2. *If $a, b \geq 0$ and $0 < \lambda < 1$, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b.$$

Proof. Assume $a, b, > 0$ and take logs:

$$\lambda \log(a) + (1_\lambda) \log(b) \leq \log(\lambda a + (1-\lambda) b)$$

by the convexity of log. □

Lemma 1.3 (Hölder's inequality)**.** *Let $1 < p < \infty$, and define $q \in (1, \infty)$ by $p^{-1} + q^{-1} = 1$. If $f, g : X \to \mathbb{C}$ are measurable, then*

$$\|fg\| \leq \|f\|_p \|g\|_q.$$

In particular, if $f \in L^p$ and $g \in L^q$, then $f, g \in L^1$. Equality holds if and only if $\alpha |f|^p = \beta |g|^q$ for some $\alpha, \beta \in \mathbb{C}$ not both zero.

Remark 1.1. In the statement of this lemma, q is called the **conjugate exponent** of p.

Proof. We may assume $9 < \|f\|_p, \|g\|_q < \infty$. The inequality holds for γf and λg for constants γ, λ iff it holds for f, g, so we may replace f, g by $f/\|f\|_p$ and $g/\|g\|_q$.[2] Let $\lambda = 1/p$, $1 - \lambda = 1/q$, and apply the previous inequality:

$$|f(x)g(x)| = (f(x)^p)^\lambda (|g(x)|)^{1-\lambda} \leq \lambda |f(x)|^p + (1-\lambda)|g(x)|^q.$$

Now integrate with respect to μ on both sides.

The equality case, after we do the reduction, is the case where $f^p = g^q$. □

Lemma 1.4 (Minkowki's inequality)**.** *If $1 \leq p < \infty$, then $\| \cdot \|_p$ satisfies the triangle inequality.*

Proof. Assume $f > 1$, and let $r, g \in L^-$. Then

$$|f+g|^p \leq (|f| + |g|)|f+g|^{p-1} = \int |f||f+g|^{p-1} \, d\mu + \int |g||f+g|^{p-1} \, d\mu$$

Apply Hölder's inequality again,

$$\leq \|f_p\| \| |f+g|^{p-1} \|_q + \|g\|_p \| |f+g|^{p-1} \|_q.$$

We can now check, using $q = p/(p-1)$, that

$$\| |f+g|^p \|_q = \left(\int |f+g|^{p(q-1)} \, d\mu \right)^{(p-1)/p} = \left(\int |f+g|^p \, d\mu \right)^{(p-1)/p} = \|f+g\|_p^{p-1}. \quad \square$$

[2]Terence Tao says that in situations like this, we have just "spent a symmetry." In this case, it is a symmetry under scalar multiplication.

Corollary 1.1. *Let* $1 \le p < \infty$.$(L^p, \|\cdot\|_p)$ *is a normed space.*

Proof. We have shown that L^p is a vector space, and $\|\cdot\|_p$ satisfies the triangle inequality. The L^p norm is homogeneous of order 1, and if $\|f_p\| = 0$, then $\int |f|^p = 0$, which makes $f = 0$ μ-a.e. □

1.3 Convergence in L^p spaces

Theorem 1.1. *Let* $1 \le p < \infty$. *Then* L^p *is a Banach space.*

Proof. Assume $\sum_n f_n$ is absolutely convergent in L^p; i.e. $\sum_n \|f_n\|_p < \infty$. Let $G_n = \sum_{i=1}^n |f_i| \in L^p$. It satisfies $\|G_n\|_p \le \sum_{i=1}^n \|f_i\|_p$ and $G_n(x) \uparrow G(x)$, where G is measurable and $[0,\infty]$-valued. By the monotone convergence theorem, $\|G_n\|_p \uparrow \|G\|_p$. Since $\|G_n\|_p \le \sum_n \|f_n\|_p$, $\|G\|_p \le \sum_n \|f_n\|_p$. So G is finite a,e,, and $G \in L^p$. So $\sum_n f_n(x)$ is absolutely convergent whenever $G(x) < \infty$ (i.e. a.e.). Let's call this pointwise limit f. $|f|^p \le |g|^p$ a.e. so $|f^p| \in L^1$; that is, $|f| \in L^p$. Finally,

$$\left| f - \sum_{i=1}^n f_i \right|^p \le 2^p |G|^p \in L^1.$$

By the dominated convergence theorem,

$$\int |f - \sum_{i=1}^n f_i|^p \, d\mu \xrightarrow{n\to\infty} 0,$$

so

$$\left(\int |f - \sum_{i=1}^n f_i|^p \, d\mu \right)^{1/p} \xrightarrow{n\to\infty} 0.$$ □

Proposition 1.1. *For* $1 \le p < \infty$, *the set of integrable simple functions is dense in* L^p.

Proof. Let $f \in L^p$. There exist complex-valeued simple functions $(\psi_n)_n$ such that $\psi_n \to f$ a.e. and $|\psi_1| \le |\psi_2| \le \cdots \le |f|$. Then $|f - \psi_n|^p \le 2|f|^p \in L^1$, so $\|f - \psi_n\| \to 0$ by the dominated convergence theorem. □

Corollary 1.2. *Let* m *be Lebesgue measure on* $\mathbb{R}^d$. *Then the collection of functions* $f \in C(\mathbb{R}^d, \mathbb{C})$ *with bounded support is dense in* $L^p(m)$.

1.4 L^∞ spaces

Definition 1.3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be measurable. The L^∞ **norm** or **essential supremum** is

$$\|f\|_\infty = \operatorname*{ess\,sup}_x |f(x)| = \inf\{a \ge 0 : \mu(\{|f| > a\}) = 0\}.$$

Definition 1.4. $L^\infty(\mu)$ is the set of equivalence classes of functions f with $\|f\|_\infty < \infty$, under the equivalence realtion of a.e. equality.

Theorem 1.2. *L^∞ has the following properties:*

1. *For all f, g, $\|fg\|_q \leq \|f\|_1 \|g\|_\infty$.*
2. *$\|\cdot\|_\infty$ is a norm.*
3. *L^∞ is complete.*
4. *$f_n \to f$ in L^∞ iff there exists $E \in \mathcal{M}$ with $\mu(E^c) = 0$ such that $f_n|_E \to f|_E$ uniformly.*
5. *The set of simple functions (not necessarily integrable) is dense in L^∞.*